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Phase splitting for periodic Lie systems

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Abstract

In the context of the Floquet theory, using a variation of parameter argument, we show that the logarithm of the monodromy of a real periodic Lie system with appropriate properties admits a splitting into two parts called dynamic and geometric phases. The dynamic phase is intrinsic and linked to the Hamiltonian of a periodic linear Euler system on the co-algebra. The geometric phase is represented as a surface integral of the symplectic form of a co-adjoint orbit.

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1. Introduction

The so-called Lie systems, representing a special sort of (generally nonlinear) time-dependent dynamics on G-spaces, are of great interest in the integrability theory of non-autonomous systems of ODEs as well as in various physical applications (see, for example, [3–7] and references therein). In this paper, we are interested in periodic Lie systems in the context of the Floquet theory and the phase splitting problem. The original motivation for this problem came from quantum mechanics [1] and then, for classical integrable systems, the phase phenomena was studied in [11]. A general geometric approach for computing the geometric and dynamic phases for Hamiltonian systems with symmetries was developed in [14]. Our goal is to give a version of phase splitting for periodic Lie systems which is based on a variation of parameter argument [10, 18]. First, we observe that the geometric and dynamic phases (as elements of a Lie algebra) are naturally defined for every smooth one-parameter family of uniformly reducible periodic Lie systems containing the trivial system. Then, we show that an individual periodic Lie system, which is reducible and contractible, is included in such a family and hence, the logarithm of its monodromy is the sum of two parts called the dynamic and geometric phases. The dynamical phase is defined in an intrinsic way and linked to the time-dependent Hamiltonian of a periodic linear Euler system on the co-algebra. The geometric phase is given as a surface integral of the symplectic form of a co-adjoint orbit. This result can be viewed as a generalization of 'scalar' formulae for the dynamic and geometric phases of cyclic solutions to periodic linear Euler systems [10, 18]. Moreover, in the case when a Lie system comes from a simple mechanical system, we show that our result agrees with known phase formulae in the reconstruction theory for Hamiltonian systems with symmetries [2, 14].

The question on the phase splitting naturally appears in the spectral problems for quantum systems in the semiclassical approximation [12, 13]. Here, the classical geometric phase defines a correction to the Bohr–Sommerfeld quantization rule and the dynamical phase gives an excitation of the classical energy level. In this context, one of the possible applications of our results is related to the computation of semiclassical spectra of spin-like quantum systems whose classical limits are just periodic Lie systems.

2. The Floquet theory for periodic Lie systems

The classical Floquet theory for linear periodic systems [19] can naturally be extended to the nonlinear case. Suppose we start with a time-periodic dynamical system on a smooth manifold M:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = X_t(x), \qquad x \in M. \tag{2.1}$$

Here, X_t is a smooth time-dependent vector field on M which is 2π-periodic in t, $X_{t+2\pi}(x) = X_t(x)$. We assume that X_t is complete. Let $\mathrm{Fl}^t : M \to M$ be the flow of X_t:

$$\frac{\mathrm{dFl}^t(x)}{\mathrm{d}t} = X_t \circ \mathrm{Fl}^t(x), \qquad \mathrm{Fl}^0 = \mathrm{id}_M.$$

From the periodicity condition and standard arguments, it follows that

$$\mathrm{Fl}^{t+2\pi} = \mathrm{Fl}^t \circ \mathrm{Fl}^{2\pi} \qquad \forall\, t \in \mathbb{R}. \tag{2.2}$$

The diffeomorphism $\mathrm{Fl}^{2\pi} \in \mathrm{Diff}(M)$ is called the *monodromy* of (2.1). System (2.1) is said to be *reducible* if there exists a time-dependent diffeomorphism on M, 2π-periodic in t, which transforms the system into an autonomous one. Observe that the reducibility property is equivalent to the *Floquet representation* for the flow of X_t:

$$\mathrm{Fl}^t = P^t \circ \Xi^t, \tag{2.3}$$

where $\Xi^t : M \to M$ is a one-parameter group of diffeomorphisms, $\Xi^{t+\tau} = \Xi^t \circ \Xi^\tau$, and $P^t : M \to M$ is a time-dependent diffeomorphism such that $P^{t+2\pi} = P^t$. Indeed, if there exists a 2π-periodic change of variables

$$x \mapsto y = (P^t)^{-1}(x), \qquad P^0 = \mathrm{id}_M, \tag{2.4}$$

transforming (2.1) to the system

$$\frac{\mathrm{d}y}{\mathrm{d}t} = Y(y), \tag{2.5}$$

with time-independent vector field

$$Y = \frac{\mathrm{d}(P^t)^{-1}}{\mathrm{d}t} \circ P^t + (P^t)_*^{-1} X_t, \tag{2.6}$$

then (2.3) holds, where the one-parameter group Ξ^t is defined as the flow of Y. Conversely, if the flow admits decomposition (2.3) for some Ξ^t and P^t, then the 2π-periodic transformation (2.4) reduces system (2.1) to the form (2.5) with $Y(y) = \frac{\mathrm{d}}{\mathrm{d}t}\big|_{t=0} \Xi^t(y)$. We obtain the following reducibility criterion: time-periodic system (2.1) is reducible if and only if there exists a one-parameter group of diffeomorphisms $\Xi^t : M \to M$ such that

$$\mathrm{Fl}^{2\pi} = \Xi^{2\pi}. \tag{2.7}$$

Therefore, as in the linear case, the reducibility of system (2.1) is completely controlled by its monodromy.

It is also useful to introduce the notion of relative reducibility. Let $\mathcal{G} \subset \mathrm{Diff}(M)$ be a subgroup of diffeomorphisms on M. Assume that the flow Fl^t takes values in $\mathcal{G}$, that is, $\mathrm{Fl}^t \in \mathcal{G}$ for all $t \in \mathbb{R}$. Then, we say that system (2.1) is *reducible relative to* $\mathcal{G}$ (or, shortly, $\mathcal{G}$*-reducible*), if one can choose a reducibility, time-dependent, diffeomorphism P^t in (2.4), (2.5) to be a 2π-periodic curve in the subgroup $\mathcal{G}$. The criterion above is modified as follows: the $\mathcal{G}$-reducibility of system (2.1) is equivalent to the existence of a one-parameter subgroup $\{\Xi^t\}$ in $\mathcal{G}$ which passes through the monodromy at $t = 2\pi$.

In general, the question on the embedding of the monodromy into a one-parameter group of diffeomorphisms is a difficult task. Our point is to discuss this problem for a special class of time-dependent dynamical systems on G-spaces, namely the *Lie systems* [3].

Suppose that the manifold M is endowed with a smooth left action $\Phi : G \times M \to G$ of a real connected Lie group G. For every $g \in G$, denote by $\Phi_g : M \to M$ the diffeomorphism given by $\Phi_g(x) = \Phi(g, x)$ for all $x \in M$. Fixing $x \in M$, we also define the smooth mapping $\Phi^x : G \to M$ letting $\Phi^x(g) = \Phi(g, x)$. Let $\mathfrak{g}$ be the Lie algebra of G. By a *periodic Lie system* on M associated with the G-action we mean the following non-autonomous system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = T_e\Phi^x(\phi(t)), \qquad x \in M, \tag{2.8}$$

where $T_e\Phi^x : \mathfrak{g} \to T_xM$ is the tangent map of Φ^x at the identity element e and $\mathbb{R} \ni t \mapsto \phi(t) \in \mathfrak{g}$ is a smooth 2π-periodic curve in the Lie algebra, i.e. $\phi(t + 2\pi) = \phi(t)$. The vector field $X_t(x) = T_e\Phi^x(\phi(t))$ of this system is represented as a linear combination of the infinitesimal generators of the G-action with time-periodic coefficients. In general, X_t is not G-invariant but the trajectories of (2.8) belong to the orbits of the G-action. Let $L_g : G \to G$ and $R_g : G \to G$ denote the left and right translations by an element $g \in G$, respectively. One can associate with (2.8) the following non-autonomous system in G

$$\frac{\mathrm{d}g}{\mathrm{d}t} = T_eR_g(\phi(t)), \qquad g \in G. \tag{2.9}$$

System (2.9) is complete because its vector field is time-periodic and right invariant. Since the right-invariant vector fields are the infinitesimal generators of the action of G on itself by the left translations, system (2.9) can also be viewed as a periodic Lie system associated with this left G-action. Consider the solution $\mathbb{R} \ni t \mapsto f(t) \in G$ to the initial value problem

$$\frac{\mathrm{d}f}{\mathrm{d}t} = T_eR_f(\phi(t)), \qquad f(0) = e. \tag{2.10}$$

Then, we have the following relationship between the flow Fl^t of system (2.8) and the solution $f(t)$ of system (2.10):

$$\mathrm{Fl}^t = \Phi_{f(t)}. \tag{2.11}$$

In particular, the flow of system (2.9) is given by $g \mapsto L_{f(t)}g$. In analogy with the linear case, we will call $f(t)$ the *fundamental solution* of the periodic Lie system (2.9) on G. In terms of the fundamental solution, the property (2.2) for the flow of (2.9) reads $f(t + 2\pi) = f(t) \cdot m$. Here, $m := f(2\pi) \in G$ is said to be the *monodromy element* of (2.9). It follows from (2.11) that the monodromy of system (2.8) is represented as $\mathrm{Fl}^{2\pi} = \Phi_m$. Let $\mathcal{G}_{G,\Phi} = \{\Phi_g, g \in G\}$ be the subgroup of diffeomorphisms on M generated by the G-action. Then, formula (2.11) says that $\mathrm{Fl}^t \in \mathcal{G}_{G,\Phi}$ and hence one can talk on the reducibility of (2.8) relative to the group $\mathcal{G}_{G,\Phi}$.

Assume that the monodromy element m lies in the image of the exponential map $\exp : \mathfrak{g} \to G$:

$$m = \exp k, \tag{2.12}$$

for a certain $k \in \mathfrak{g}$. This implies that the Floquet representation for the fundamental solution reads

$$f(t) = L_{p(t)}\left(\exp\left(\frac{tk}{2\pi}\right)\right), \tag{2.13}$$

where $t \mapsto p(t)$ is a 2π-periodic curve in G with $p(0) = e$. It follows from here and (2.11) that the monodromy $\mathrm{Fl}^{2\pi}$ satisfies condition (2.3) for the one-parameter group of diffeomorphisms $\Xi^t = \Phi_{\exp(\frac{tk}{2\pi})}$ and hence system (2.8) is $\mathcal{G}_{G,\Phi}$-reducible. Therefore, under the 2π-periodic change of variables (2.4) with

$$P^t = \mathrm{Fl}^t \circ \Phi_{\exp(-\frac{tk}{2\pi})} = \Phi_{p(t)},$$

system (2.8) is transformed to the autonomous Lie system of the form $\mathrm{d}y/\mathrm{d}t = T_e\Phi^y\left(\frac{k}{2\pi}\right)$. Condition (2.12) also becomes necessary for the reducibility under a natural assumption on the G-action. We have the following criterion: if the action Φ of G on M is *faithful*, then property (2.12) is a sufficient and necessary condition for the $\mathcal{G}_{G,\Phi}$-reducibility of system (2.8). In particular, periodic Lie system (2.9) is reducible relative to the group of left translations on G if and only if (2.12) holds.

One can also show that if the G-action is not faithful, then the criterion of the $\mathcal{G}_{G,\Phi}$-reducibility for system (2.8) leads to the following representation for the monodromy element:

$$m = m_0 \cdot \exp k, \tag{2.14}$$

for a certain element m_0 in the kernel of the homomorphism $g \mapsto \Phi_g$. In this case, system (2.9) in G is not necessarily reducible. In terms of the monodromy of (2.8), reducibility condition (2.14) reads $\mathrm{Fl}^{2\pi} = \Phi_{\exp k}$.

Consider the following important class of periodic Lie systems associated with linear representations of G. Let $\mathrm{Ad} : G \times \mathfrak{g} \to \mathfrak{g}$ be the adjoint action of the Lie group G on its Lie algebra, $\mathrm{Ad}_g = T_g R_{g^{-1}} \circ T_e L_g$. Taking $\Phi = \mathrm{Ad}$ and $M = \mathfrak{g}$ for (2.8), we get the following Lie system:

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathrm{ad}_{\phi(t)}x, \qquad x \in \mathfrak{g}, \tag{2.15}$$

which is called a *periodic linear Euler system* on $\mathfrak{g}$. Here, $\mathrm{ad}_\phi : \mathfrak{g} \to \mathfrak{g}$ is the adjoint operator, $\mathrm{ad}_\phi y = [\phi, y]$. It follows from (2.11) that the flow of (2.15) is $\mathrm{Fl}^t = \mathrm{Ad}_{f(t)}$, where $f(t)$ is the fundamental solution in (2.10). Therefore, Fl^t takes values in the adjoint group $\mathrm{Ad}\,\mathfrak{g}$ of the Lie algebra $\mathfrak{g}$, which is generated by the elements $\exp(\mathrm{ad}_z)$, for $z \in \mathfrak{g}$. Since G is connected, the kernel of the adjoint representation $g \mapsto \mathrm{Ad}_g$ coincides with the centre $Z(G)$ of the Lie group. Then, the $\mathrm{Ad}\,\mathfrak{g}$-reducibility of linear Euler equation (2.15) is equivalent to representation (2.14) for some $m_0 \in Z(G)$ and $k \in \mathfrak{g}$. In terms of the monodromy $\mathfrak{M} = \mathrm{Ad}_m$ of (2.15), the reducibility criterion says that $\mathfrak{M} = \exp(\mathrm{ad}_k)$.

Remark that the reducibility condition (2.12) automatically holds in the case when G belongs to the class of *exponential* Lie groups which includes, for example, the Lie groups of *compact type* [8, 9, 17]. If the Lie group G is not exponential, then one can apply the following criterion. Assume that the monodromy element m is *regular* and the isotropy subgroup $G_m = \{\alpha \in G \mid \alpha \cdot m = m \cdot \alpha\}$ is *connected*. Then, it follows that G_m is *Abelian* [9] and hence, (2.12) holds.

Example 2.1. Let $G = SO(3)$ be the compact Lie group of all orthogonal 3×3 matrices g with $\det g = 1$ and $\mathfrak{g} = \mathfrak{so}(3)$ its Lie algebra of skew-symmetric matrices. A periodic Lie system on $SO(3)$ is written as

$$\frac{\mathrm{d}g}{\mathrm{d}t} = (\Lambda \circ w(t)) \cdot g, \tag{2.16}$$

where $t \mapsto w(t) \in \mathbb{R}^3$ is a 2π-periodic vector function and $\Lambda \circ x$ denotes the matrix of the cross product in $\mathbb{R}^3$, $(\Lambda \circ x)y = x \times y$. Under the identification of $\mathfrak{so}(3)$ with $\mathbb{R}^3$, the corresponding periodic linear Euler system takes the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = w(t) \times x, \qquad x \in \mathbb{R}^3. \tag{2.17}$$

Since the monodromy element $m \in SO(3)$ of (2.16) is a rotation in $\mathbb{R}^3$, we have $m = \exp \Lambda \circ \nu$ for $\nu \in \mathbb{R}^3$. Therefore, systems (2.16) and (2.17) are reducible.

The next example is related to the reducibility of periodic linear Hamiltonian systems on $\mathbb{R}^2$ [19].

Example 2.2. Consider the special linear group $G = SL(2; \mathbb{R})$ of all real 2×2 matrices with determinant 1. The corresponding Lie algebra $\mathfrak{g} = \mathfrak{sl}(2; \mathbb{R})$ consists of traceless 2×2 matrices. A periodic Lie system in $SL(2; \mathbb{R})$ is of the form

$$\frac{\mathrm{d}g}{\mathrm{d}t} = \begin{bmatrix} a_1(t) & a_2(t) \\ a_3(t) & -a_1(t) \end{bmatrix} \cdot g, \tag{2.18}$$

where $a_i(t)$ $(i = 1, 2, 3)$ are 2π-periodic, real functions. It is well known [8, 9] that the exponential map for $SL(2; \mathbb{R})$ is not surjective. The monodromy element $m \in SL(2; \mathbb{R})$ of (2.18) has the representation $m = \pm\exp k$, for $k = \left[\begin{smallmatrix} k_1 & k_2 \\ k_3 & -k_1 \end{smallmatrix}\right]$. Moreover, m is in the image of the exponential map if and only if $\operatorname{tr} m > -2$ or $m = -I$. In the opposite case, system (2.18) is not reducible. Identifying $\mathfrak{sl}(2, \mathbb{R})$ with $\mathbb{R}^3$, we can write the periodic linear Euler system associated with (2.18) in the form

$$\frac{\mathrm{d}x}{\mathrm{d}t} = \mathcal{I}(w(t) \times x), \qquad x \in \mathbb{R}^3, \tag{2.19}$$

where $\mathcal{I} = \mathrm{diag}(1, 1, -1)$ and $w(t) = (2a_1(t), -a_2(t) - a_3(t), a_2(t) - a_3(t))$. The kernel of the adjoint representation of $SL(2; \mathbb{R})$ is the two element group $\{I, -I\}$. The adjoint group of $\mathfrak{sl}(2; \mathbb{R})$ is isomorphic to the Lorentz group $SO^+(2, 1)$ which is exponential [8]. Therefore, linear Euler system (2.19) is reducible since for its monodromy $\mathfrak{M} \in SO^+(2, 1)$ we have $\mathfrak{M} = \exp \mathcal{I}(\Lambda \circ v)$, for $v = (2k_1, -k_2 - k_3, k_2 - k_3) \in \mathbb{R}^3$.

3. The mapping D

Denote by $C_e^\infty(\mathbb{R}, G)$ the set of all smooth curves $\alpha(t) : \mathbb{R} \to G$ in the Lie group with $\alpha(0) = e$, and $C^\infty(\mathbb{R}, \mathfrak{g})$ the set of all smooth curves in the Lie algebra $\mathfrak{g}$. Introduce the mapping $\mathrm{D} : C_e^\infty(\mathbb{R}, G) \to C^\infty(\mathbb{R}, \mathfrak{g})$ given by

$$\mathrm{D}\alpha(t) := T_{\alpha(t)} R_{\alpha(t)^{-1}} \left(\frac{\mathrm{d}\alpha}{\mathrm{d}t}(t) \right) \in \mathfrak{g}.$$

Then, in terms of D, equation (2.10) for the fundamental solution $f(t)$ is rewritten as follows:

$$\mathrm{D}f = \phi. \tag{3.1}$$

Moreover, for any $a \in \mathfrak{g}$ and $\alpha, \beta \in C_e^\infty(\mathbb{R}, G)$, the following identities hold [9]:

$$\mathrm{D}(\exp ta) = a, \tag{3.2}$$

$$\mathrm{D}\alpha^{-1}(t) = -\mathrm{Ad}_{\alpha^{-1}(t)}\mathrm{D}\alpha(t), \tag{3.3}$$

$$\mathrm{D}(L_\alpha\beta)(t) = \mathrm{D}\alpha(t) + \mathrm{Ad}_{\alpha(t)}\mathrm{D}\beta(t), \tag{3.4}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathrm{Ad}_{\alpha(t)} = \mathrm{ad}_{\mathrm{D}\alpha(t)} \circ \mathrm{Ad}_{\alpha(t)}. \tag{3.5}$$

Let σ be a parameterized surface in G given by a smooth map $\mathbb{R}^2 \ni (s,t) \mapsto \sigma(s,t) \in G$. Denote by D_s and D_t the mappings which act on the s-parameter and t-parameter families of curves σ_t and σ_s associated with σ, respectively,

$$\mathrm{D}_s\sigma(s,t) := \mathrm{D}\sigma_t(s) \equiv T_{\sigma(s,t)}R_{\sigma(s,t)^{-1}}\left(\frac{\partial\sigma(s,t)}{\partial s}\right),$$

$$\mathrm{D}_t\sigma(s,t) := \mathrm{D}\sigma_s(t) \equiv T_{\sigma(s,t)}R_{\sigma(s,t)^{-1}}\left(\frac{\partial\sigma(s,t)}{\partial t}\right).$$

One can show that the relationship between these two mappings is given by the 'zero curvature'-type equation:

$$\frac{\partial \mathrm{D}_t\sigma}{\partial s} - \frac{\partial \mathrm{D}_s\sigma}{\partial t} + [\mathrm{D}_t\sigma, \mathrm{D}_s\sigma] = 0. \tag{3.6}$$

4. Dynamic and geometric phases

Suppose we start with a family of periodic Lie systems in G of the form (2.9) associated with a s-parameter family $\{\phi_s\}$ of closed curves in $\mathfrak{g}$ given by a C^∞ mapping $[0,1]\times\mathbb{R} \ni (s,t) \mapsto \phi(s,t) \in \mathfrak{g}$ with

$$\phi(s, t+2\pi) = \phi(s,t), \tag{4.1}$$

$$\phi(0,t) = 0. \tag{4.2}$$

Let $f(s,t)$ be the parameter-dependent fundamental solution:

$$\frac{\mathrm{d}f(s,t)}{\mathrm{d}t} = T_eR_{f(s,t)}(\phi(s,t)), \qquad f(s,0) = e. \tag{4.3}$$

It is clear that $f(s,t)$ is smooth in both variables s and t. Moreover, $f(0,t) = e$ because of (4.2). Assume that the family of periodic Lie systems is uniformly reducible, that is, for every $s \in [0,1]$, the monodromy element has the representation

$$m(s) = f(s, 2\pi) = \exp k(s), \tag{4.4}$$

for a certain $k(s) \in \mathfrak{g}$, smoothly varying in s and such that $k(0) = 0$. Consider the G-valued function

$$p(s,t) = L_{f(s,t)}\exp\left(-\frac{tk(s)}{2\pi}\right), \tag{4.5}$$

with properties

$$p(s, t+2\pi) = p(s,t), \tag{4.6}$$

$$p(s,0) = e, \tag{4.7}$$

$$p(0,t) = e. \tag{4.8}$$

Applying the mapping D_t to both sides of (4.5) and by using (3.1), (3.2) and (3.4), we derive the identity

$$\frac{k(s)}{2\pi} = \mathrm{Ad}_{p(s,t)^{-1}}(\phi(s,t) - \mathrm{D}_t p(s,t)).$$

Integrating this equality in t over $[0, 2\pi]$ gives

$$k(s) = \int_0^{2\pi} \mathrm{Ad}_{p(s,t)^{-1}}(\phi(s,t))\,\mathrm{d}t - \int_0^{2\pi} \mathrm{Ad}_{p(s,t)^{-1}}(\mathrm{D}_t p(s,t))\,\mathrm{d}t. \tag{4.9}$$

By using (3.3), (3.6) and (4.8), we compute the second term in (4.9):

$$\begin{aligned}\int_0^{2\pi} \mathrm{Ad}_{p(s,t)^{-1}}(\mathrm{D}_t p(s,t))\,\mathrm{d}t &= -\int_0^{2\pi} \mathrm{D}_t p^{-1}(s,t)\,\mathrm{d}t \\ &= -\int_0^{2\pi}\int_0^s \frac{\partial}{\partial u}(\mathrm{D}_t p^{-1}(u,t))\,\mathrm{d}u\,\mathrm{d}t + \int_0^{2\pi} \mathrm{D}_t p^{-1}(0,t)\,\mathrm{d}t \\ &= \int_0^s (-\mathrm{D}_u p^{-1}(u,2\pi) + \mathrm{D}_u p^{-1}(u,0))\,\mathrm{d}u \\ &\quad - \int_0^{2\pi}\int_0^s [\mathrm{D}_u p^{-1}(u,t), \mathrm{D}_t p^{-1}(u,t)]\,\mathrm{d}u\,\mathrm{d}t.\end{aligned}$$

The first summand in the last formula vanishes because of (4.6) and (4.7). On summarizing, we obtain the following result.

Theorem 4.1. *For every* $s \in [0,1]$, *the* log *phase* $k(s)$ *in (4.4) has the decomposition*

$$k(s) = k_{\mathrm{dyn}}(s) + k_{\mathrm{geom}}(s), \tag{4.10}$$

where

$$k_{\mathrm{dyn}}(s) = \int_0^{2\pi} \mathrm{Ad}_{p^{-1}(s,t)}\phi(s,t)\,\mathrm{d}t, \tag{4.11}$$

$$k_{\mathrm{geom}}(s) = \int_0^{2\pi}\int_0^s [\mathrm{D}_u p^{-1}(u,t), \mathrm{D}_t p^{-1}(u,t)]\,\mathrm{d}t\,\mathrm{d}u. \tag{4.12}$$

The components $k_{\mathrm{dyn}}(s)$ and $k_{\mathrm{geom}}(s)$ will be called the *dynamic* and *geometric phases* of the family of periodic Lie systems, respectively. Now, let us give interpretations of $k_{\mathrm{dyn}}(s)$ and $k_{\mathrm{geom}}(s)$ in terms of the Poisson geometry and Hamiltonian dynamics on the dual space (the co-algebra) $\mathfrak{g}^*$ of $\mathfrak{g}$. Let $\Phi : G \times \mathfrak{g}^* \to \mathfrak{g}^*$ be the left action of G on the co-algebra $\mathfrak{g}^*$ given by $\Phi_g = \mathrm{Ad}^*_{g^{-1}}$, where $\mathrm{Ad}^*_\alpha : \mathfrak{g}^* \to \mathfrak{g}^*$ is the co-adjoint action of the Lie group. Then, Lie system (2.8) associated with such action and the function $\phi = \phi(s,t)$ gives the family of periodic linear Euler systems on $\mathfrak{g}^*$:

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -\mathrm{ad}^*_{\phi(s,t)}\xi, \qquad \xi \in \mathfrak{g}^*. \tag{4.13}$$

It follows from (2.11) that the flow (the fundamental solution) of (4.13) is given by $\mathrm{Fl}^t = \mathrm{Ad}^*_{f^{-1}(s,t)}$, where $f(s,t)$ is the G-valued fundamental solution in (2.10). Taking into account (2.11), we get that the monodromy of (4.13) is of the form $\mathrm{Fl}^{2\pi} = \exp(-\mathrm{ad}^*_{k(s)})$. For every s, system (4.13) represents a time-dependent Hamiltonian system relative to the 'plus' Lie–Poisson bracket on $\mathfrak{g}^*$ [15] and the function $H_t(\xi) = -\langle \xi, \phi(s,t)\rangle$. Here, we denote by $\langle\, , \rangle$ the pairing between vectors and covectors. Pick a point $\mu \in \mathfrak{g}^*$. Then, in terms of the time-dependent Hamiltonian $H_t : \mathfrak{g}^* \to \mathbb{R}$, we have the following representation for the dynamical phase:

$$\langle \mu, k_{\mathrm{dyn}}(s)\rangle = -\int_0^{2\pi} H_t(\mathrm{Ad}^*_{p^{-1}(s,t)}\mu)\,\mathrm{d}t. \tag{4.14}$$

Here, $p(s, t)$ is defined by (4.5). Let $\mathcal{O} \subset \mathfrak{g}^*$ be the co-adjoint orbit passing through the point μ. Fix $s \in [0, 1]$ and consider the oriented cylinder $\mathcal{C}^2_s = [0, s] \times \mathbb{S}^1$ with coordinates $(s, u \bmod 2\pi)$. Define the C^∞ mapping $F : \mathcal{C}^2_s \to \mathfrak{g}^*$ by $F(u, t) = \mathrm{Ad}^*_{p^{-1}(u,t)}\mu$. It is clear that the image of $\mathcal{C}^2_s$ under F lies in the co-adjoint orbit, $F\left(\mathcal{C}^2_s\right) \subset \mathcal{O}$. Moreover, $F(\{0\} \times \mathbb{S}^1) = \mu$. Let $\omega_{\mathcal{O}}$ be the symplectic form (the Kirillov form) on $\mathcal{O}$ which is given by

$$\omega_{\mathcal{O}}(\mathrm{ad}^*_x\eta, \mathrm{ad}^*_y\eta) = \langle \eta, [x, y]\rangle, \tag{4.15}$$

for $\eta \in \mathcal{O}$ and $x, y \in \mathfrak{g}$. Taking into account properties (3.3) and (3.5), we compute

$$\frac{\partial F(u,t)}{\partial t} = -\mathrm{ad}^*_{\mathrm{D}_t p(u,t)} F(u, t),$$
$$\frac{\partial F(u,t)}{\partial u} = -\mathrm{ad}^*_{\mathrm{D}_u p(u,t)} F(u, t).$$

Putting these formulae into (4.15) and using again (3.3), we obtain

$$\begin{aligned}\omega_{\mathcal{O}}\left(\frac{\partial F}{\partial u}, \frac{\partial F}{\partial t}\right) &= \langle F(u, t), [\mathrm{D}_u p(u, t), \mathrm{D}_t p(u, t)]\rangle \\ &= \langle \mathrm{Ad}^*_{p^{-1}(u,t)}\mu, [\mathrm{D}_u p(u, t), \mathrm{D}_t p(u, t)]\rangle \\ &= \langle \mu, [\mathrm{D}_u p^{-1}(u, t), \mathrm{D}_t p^{-1}(u, t)]\rangle.\end{aligned}$$

Comparing this with (4.12) leads to the following representation for the geometric phase:

$$\langle \mu, k_{\mathrm{geom}}(s)\rangle = \int_{\mathcal{C}^2_s} F^*\omega_{\mathcal{O}}. \tag{4.16}$$

If the mapping F is regular, then the right-hand side of (4.16) is the symplectic area of the oriented surface $\Sigma_s = F\left(\mathcal{C}^2_s\right)$ in $\mathcal{O}$ whose boundary is the loop $\gamma_s = \{\xi = \mathrm{Ad}^*_{p^{-1}(u,s)}\mu\}$. Note that formulae (4.14) and (4.16) remain also valid if $\mu = \mu(s)$ varies smoothly with s and lies at a co-adjoint orbit $\mathcal{O}$ for all $s \in [0, 1]$. In the case when $\mathrm{ad}^*_{k(s)}\mu(s) = 0$, the parameterized curves γ_s are periodic solutions of linear Euler system (4.13) and the values $\langle \mu(s), k_{\mathrm{dyn}}(s)\rangle$ and $\langle \mu(s), k_{\mathrm{geom}}(s)\rangle$ correspond to the dynamic and geometric parts in the splitting of Floquet exponents of cyclic solutions of (4.13) (for more details, see [10, 18]).

Remark 4.2. If instead of (4.1) we have

$$\phi(s, t + T(s)) = \phi(s, t)$$

for a certain smooth positive function $T(s)$, then the geometric phase of the corresponding family of periodic Lie systems is given by (4.16) and the formula for the dynamic phase is modified as follows:

$$\langle \mu, k_{\mathrm{dyn}}(s)\rangle = -\int_0^{T(s)} H_t(\mathrm{Ad}^*_{p^{-1}(s,t)}\mu)\,\mathrm{d}t. \tag{4.17}$$

Here, $p(s, t) = L_{f(s,t)} \exp\left(-\frac{tk(s)}{T(s)}\right)$ is a $T(s)$-periodic in t for each s.

Now, using the above results, we introduce dynamical and geometric phases for an *individual* periodic Lie system (2.9) satisfying reducibility condition (2.12) for a certain element $k \in \mathfrak{g}$. Consider the loop $\Gamma : t \mapsto p(t)$ in G, based at e, where $p(t) = f(t)\exp(-tk/2\pi)$ is the 2π-periodic, G-valued function corresponding to the periodic factor in the Floquet representation. Then, Γ depends on the choice of k in (2.12), but it is easy to see that the homotopy class $[\Gamma]$ of Γ in $\pi_1(G)$ is independent of any such choice. Assume that $[\Gamma]$ is *trivial*. In this case, we say that the reducible periodic Lie system is *contractible*. This condition means that we can fix a smooth homotopy in G of the loop Γ to the unity e which is given by a C^∞ function $p(s, t)$ satisfying (4.8) and $p(1, t) = p(t)$. Pick an arbitrary C^∞ function $k(s)$ on [0, 1] with $k(0) = 0$ and $k(1) = k$. For example, one can put $k(s) = sk$. Then, we define

$$\phi(s,t) := \frac{1}{2\pi}\mathrm{Ad}_{p(s,t)}k(s) + \mathrm{D}_t p(s,t). \tag{4.18}$$

Clearly, this function satisfies properties (4.1), (4.2) and $\phi(1, t) = \phi(t)$. Therefore, we have proved that the original Lie system is included into a smooth family of reducible periodic Lie systems on G associated with ϕ in (4.18) which is contractible to the trivial system $\dot{g} = 0$. Applying formulae (4.14) and (4.16) to this family, we arrive at the final result.

Theorem 4.3. *Assume that a periodic Lie system*

$$\frac{\mathrm{d}g}{\mathrm{d}t} = T_eR_g(\phi(t)), \qquad g \in G,$$

is reducible, $m \in \exp(\mathfrak{g})$*, and contractible. Let p(s, t) be an arbitrary smooth homotopy of* Γ *to the identity e. Then, the monodromy element has the representation*

$$m = \exp(k_{\mathrm{dyn}} + k_{\mathrm{geom}}),$$

where the dynamic and geometric phases $k_{\mathrm{dyn}}, k_{\mathrm{geom}} \in \mathfrak{g}$ *are given by*

$$\langle \mu, k_{\mathrm{dyn}}\rangle = -\int_0^{2\pi} H_t(F(1,t))\,\mathrm{d}t, \tag{4.19}$$

$$\langle \mu, k_{\mathrm{geom}}\rangle = \int_{[01]\times\mathbb{S}^1} F^*\omega_{\mathcal{O}}, \tag{4.20}$$

for any $\mu \in \mathfrak{g}^*$*. Here* $F(s,t) = \mathrm{Ad}^*_{p^{-1}(s,t)}\mu$*,* $H_t(\xi) = -\langle \xi, \phi(s,t)\rangle$ *and* $\mathcal{O}$ *is the co-adjoint orbit through* μ*.*

Remark that the dynamic phase (4.19) is independent of the choice of a homotopy $p(s, t)$. The elements k_{dyn} and k_{geom} can also be called the dynamical and geometric phases of the periodic Lie system (2.8) in the G-space (M, G, Φ).

5. Some applications

Consider the following dynamical system in $\mathfrak{g}^* \times G$:

$$\frac{\mathrm{d}\xi}{\mathrm{d}t} = -\mathrm{ad}^*_{\frac{\delta h}{\delta \xi}}\xi, \tag{5.1}$$

$$\frac{\mathrm{d}g}{\mathrm{d}t} = T_eR_g\left(\frac{\delta h}{\delta \xi}\right), \tag{5.2}$$

where $\xi \in \mathfrak{g}^*$, $g \in G$ and $h : \mathfrak{g}^* \to \mathbb{R}$ is a C^∞ function and the element $\frac{\delta h}{\delta \xi} \in \mathfrak{g}$ is defined by the equality $\left\langle \mu, \frac{\delta h}{\delta \xi}\right\rangle = d_\xi h(\mu)$ [15]. This system comes from a G-invariant Hamiltonian system in T^*G under the identification of T^*G with $\mathfrak{g}^* \times G$ by the right translations [15]. Suppose we are given a one-parameter family of periodic trajectories $\gamma_s : t \mapsto \xi_s(t)$ of nonlinear Euler system (5.1), $\xi_s(t+T(s)) = \xi_s(t)$ which are smoothly contractible to a rest point $\xi_0(t) = \eta_0$, $d_{\eta_0}h = 0$. Putting these periodic solutions in second equation (5.2), we get a family of periodic Lie systems on G associated with the $\mathfrak{g}$-valued function

$$\phi(s,t) = \frac{\delta h(\xi_s(t))}{\delta \xi}, \tag{5.3}$$

which satisfies conditions (4.1) and (4.2). Let $f(s,t)$ be the fundamental solution of (5.2) and (5.3), and $m(s) = f(s, T(s))$ the monodromy element. Assume that the periodic orbits belong to one and the same co-adjoint orbit $\mathcal{O}$:

$$\xi_s^0 := \xi_s(0) \in \mathcal{O}, \qquad \forall s \in [0,1].$$

Remark that $\xi_s(t)$ is also a periodic solution of periodic linear Euler system (4.13) with ϕ given by (5.3). This implies that for every $s \in [0,1]$, the monodromy element $m(s)$ lies in the isotropy subgroup of the co-adjoint representation at ξ_s^0:

$$m(s) \in G_{\xi_s^0} = \left\{\alpha \in G \mid \mathrm{Ad}^*_\alpha \xi_s^0 = \xi_s^0\right\}.$$

The Lie algebra of $G_{\xi_s^0}$ is the isotropy $\mathfrak{g}_{\xi_s^0} = \left\{a \in \mathfrak{g} \,\middle|\, \mathrm{ad}^*_a \xi_s^0 = 0\right\}$. Moreover, we assume that $G_{\xi_s^0}$ is *connected* and the co-adjoint orbit $\mathcal{O}$ is *regular*. Then, $G_{\xi_s^0}$ is Abelian and exponential [9]. It follows that there exists a unique C^∞ curve $[0,1] \ni s \mapsto k(s) \in \mathfrak{g}$ such that $k(s) \in \mathfrak{g}_{\xi_s^0}$, $k(0) = 0$ and $m(s) = \exp k(s)$. Applying the results of section 4, we get that the total log phase $k(s)$ has decomposition (4.10), where $k_{\mathrm{geom}}(s)$ and $k_{\mathrm{dyn}}(s)$ are given by formulae (4.16) and (4.17), respectively. Taking into account that $\xi_s(t) = \mathrm{Ad}^*_{p^{-1}(s,t)}\xi_s^0$ and evaluating $k_{\mathrm{geom}}(s)$ at $\mu = \xi_s^0$, we get

$$\left\langle \xi_s^0, k_{\mathrm{dyn}}(s)\right\rangle = \int_0^{T(s)} \left\langle \xi, \frac{\delta h}{\delta \xi}\right\rangle\bigg|_{\xi=\xi_s(t)} \mathrm{d}t, \tag{5.4}$$

$$\left\langle \xi_s^0, k_{\mathrm{geom}(s)}\right\rangle = \int_{\Sigma_s} \omega_{\mathcal{O}}, \tag{5.5}$$

where $\Sigma_s = \bigcup_{0\leqslant s'\leqslant s} \gamma_{s'}$ is the oriented surface in $\mathcal{O}$ spanned by the periodic trajectories. In particular, if h is a quadratic form, then $\left\langle \xi, \frac{\delta h}{\delta \xi}\right\rangle = 2h(\xi)$ and

$$\left\langle \xi_s^0, k_{\mathrm{dyn}}\right\rangle = 2T(s)h\left(\xi_s^0\right). \tag{5.6}$$

Here, we use the property that h is constant along γ_s. In the context of the theory of reconstruction phases for simple mechanical systems, formulae such as (5.5) and (5.6) were derived in [2, 14]. In the case when $G = SO(3)$, these formulae lead to the well-known representations for the rigid body phases [16].

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References

[1] Berry M V 1984 Quantal phase factors accompanying adiabatic changes *Proc. R. Soc.* A **392** 45–57
[2] Blaom A D 2000 Reconstruction phases via Poisson reduction *Differ. Geom. Appl.* **12** 231–52

[3] Cariñena J F, Grabowski J and Marmo G 2000 *Lie–Scheffers Systems: A Geometric Approach* (Napoli, Italy: Bibliopolis)

[4] Cariñena J F, Grabowski J and Marmo G 2007 Superposition rules, Lie theorem and partial differential equations *Rep. Math. Phys.* **60** 237–58

[5] Cariñena J F and de Lucas J 2008 Integrability of Lie systems and some of its applications in physics *J. Phys. A: Math. Theor.* **41** 304029

[6] Cariñena J F and de Lucas J 2009 Quantum Lie systems and integrability conditions *Int. J. Geom. Methods Mod. Phys.* **6** 1235–52

[7] Cariñena J F, de Lucas J and Ramos A 2009 A geometric approach to time evolution operators of Lie quantum systems *Int. J. Theor. Phys.* **48** 1379–404

[8] Dokovic D Z and Hofmann K H 1997 The surjectivity question for the exponential function of real Lie groups: a status report *J. Lie Theory* **7** 171–99

[9] Duistermaat J J and Kolke J A 1999 *Lie Groups* (Berlin: Springer)

[10] Flores R and Vorobiev Y 2005 On dynamical and geometric phases of time-periodic linear Euler equations *Russ. J. Math. Phys.* **12** 326–49

[11] Hannay J 1985 Angle variable holonomy in adiabatic excursion of an integrable Hamiltonian *J. Phys. A: Math. Gen.* **18** 221–30

[12] Karasev M and Vorobjev Yu 1998 Adapted connections, Hamiltonian dynamics, geometric phases and quantization over isotropic submanifolds *Am. Math. Soc. Transl.* **187** 203–326

[13] Littlejohn R G 1988 Cyclic evolution in quantum mechanics and the phase of Bohr–Sommerfeld and Maslov *Phys. Rev. Lett.* **61** 2159–62

[14] Marsden J E, Montgomery R and Ratiu T S 1990 *Reduction, Symmetry, and Phases in Mechanics* (Memoirs of the American Mathematical Society) vol 88 (Providence, RI: American Mathematical Society)

[15] Marsden J E and Ratiu T S 1999 *Introduction to Mechanics and Symmetry* (New York: Springer)

[16] Montgomery R 1991 How much does a rigid body rotates? A Berry's phase from 18th century *Am. J. Phys.* **59** 394–8

[17] Moskowitz M and Sacksteder R 2003 The exponential map and differential equations on real Lie groups *J. Lie Theory* **13** 291–306

[18] Vorobjev Y M 2005 *Poisson Structures and Linear Euler Systems over Symplectic Manifolds* (American Mathmatical Society Translation, Series 2) vol 216 (Providence, RI: American Mathematical Society) pp 137–239

[19] Yakubovich V A and Starzhinskii V M 1975 *Linear Differential Equations with Periodic Coefficients* (New York: Wiley)